Exam proofs

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2) Prove $\Delta[X]_t = (\Delta X)^2$ $\quad \Delta X_t = X_t - X_{t^-}$

$[X] = X_t^2 - X_0^2 - 2\int_0^t X_{s^-} dX_s$ $\quad$ let $Y_t = \int_0^t X_{s^-} dX_s$

$\Delta[X]_t = \Delta X_t^2 - 2\Delta Y_t$ $\quad$ Recall the jump rule: $\Delta Y_t = X_{t^-} \Delta X_t$

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$= \Delta X_t^2 - 2 X_{t^-} \cdot \Delta X_t$

$= (X_t^2 - X_{t^-}^2) - 2X_{t^-} \cdot (X_t - X_{t^-})$

$= X_t^2 + X_{t^-}^2 - 2X_{t^-} \cdot X_t$

$= \underbrace{(X_t - X_{t^-})}_{\Delta X_t}^2 = (\Delta X_t)^2$

Exam questions:

2) If L is continuous and FV then the quadratic variation of L is zero.

$\Rightarrow$ if $(X_t)_{t \geq 0}$ is cont. and FV $\Rightarrow$ the quadratic variation is zero

check: $[X, Y] = \sum_{s \leq t} \Delta Y_s \Delta X_s$ $\quad$ ~~for X~~ for X or Y FV

$[X, X] = \sum_{s \leq t} (\Delta X_s)^2$ $\quad$ X FV

$\Rightarrow$ if any of the semimartingales ~~X,Y~~ X and Y cont. and if any of them is FV, then $[X, Y]_t = 0$

3) $X_t^4 = X_0^4 + \underbrace{4\int_0^t X_{s^-}^{4-1} dX_s}_{\substack{\text{stochastic integral w.r.t.} \\ \text{semimartingale} \\ \Rightarrow \text{semimartingale}}} + \underbrace{\frac{4(4-1)}{2}\int_0^t X_s^{4-2} d\overbrace{[X]_s}^{FV}}_{\substack{\text{Riemann-Stieltjes Integral, FV} \\ \Rightarrow \text{Semimartingale}}}$ $\quad$ (only for continuous)

$\uparrow$ cont

Semimartingale

or $X_t \Rightarrow$ semi-martingale $\Rightarrow f(x) = x^4 \quad f'(x) = 4x^{4-1} \quad f'' = 4(4-1)x^{4-2}$

$d(X_t^4) = f'(x)dx + \frac{1}{2} f''(x)\, d[X]_t$

$X_t^4 = X_0^4 + 4\int_0^t X_s^{4-1} dX_s + \frac{1}{2}\int_0^t 4\cdot(4-1) X_s^{4-2} d[X]_s$

②

3) a) $dW_t^2 = W_t^2$

$$W_t^2 = \underset{\uparrow =0}{W_0^2} + \int_0^t 2W_s\,dW_s + \underbrace{\frac{1}{2}\cdot\int_0^t 2\,ds}_{\frac{2}{2}\cdot t} = W_0^2 + 2\int_0^t W_s\,dW_s + t$$

b) $dW_t^3 \Rightarrow W_t^3 \Rightarrow W_t^3 = \underset{\uparrow =0}{W_0^3} + \int_0^t 3\cdot W_t^2\,dW_t + \int_0^t \cancel{3}\,3\cdot 2\cdot W_s \cdot \underset{\uparrow =s}{d[W]_s} = 3\int_0^t W_t^2\,dW_t + 6\int_0^t W_s\,ds$

$$= 3W_t^2\,dW_t + 6W_t\,dt$$

c) $dN_t^2 \Rightarrow N_t^2 \Rightarrow N_t^2 = N_0^2 + \int_0^t 2N_s\,dN_s + \int_0^t 2\,\cancel{1}\,d[N]_s =$

$[N,N]_t = [N]_t = t$

d) $(\mu t + \sigma W_t)^2 = \cancel{2}\,(\overbrace{\mu\cdot 0 + \sigma W_0}^{=0})^2 + \int_0^t 2\cdot(\mu\cdot s + \sigma W_s)\,d(\mu s + \sigma W_s) + \frac{1}{2}\int_0^t 2\cdot 1\cdot d\overbrace{[\mu s + \sigma W_s]}^{=\sigma^2 s} =$

$= \cancel{2\int_0^t \mu s + \sigma W_s\, d}$

$= 2\int_0^t (\mu s + \sigma W_s)\cdot \mu\,ds + 2\int_0^t (\mu s + \sigma W_s)\cdot\sigma\,dW_s + \frac{1}{2}\int_0^t 2\sigma^2\,ds$

$$= \int_0^t (2\mu s + 2\sigma W_s)\cdot\mu + \sigma^2\,ds + \int_0^t 2(\mu s + \sigma W_s)\cdot\sigma\,dW_s$$

4) a) $[W^2]_t =$

5) Ito-process: $X_t = X_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s$ Ito-Integral FV

a) $\int_0^t W_s\,dW_s^2 = \int_0^t W_s\,d(2\int_0^t W_s\,dW_s + s) = \int_0^t 2W_s^2\,dW_s + \int_0^t W_s\,ds$

b) $W_t^2 = \underbrace{2\int_0^t W_s\,dW_s}_{\text{Ito-Integral}} + \underset{\uparrow\text{FV}}{t}$ $\| \int_0^t 1\,ds$

c) $W_t^3 = \underbrace{3\int_0^t W_s^2\,dW_s}_{\text{Ito-Integral}} + \underbrace{3\int_0^t W_s\,ds}_{\text{FV}}$

d) $t\,W_t = \int_0^t s\,dW_s + \int_0^t W_s\,ds + \underbrace{[t,W]_t}_{=0} = \underbrace{\int_0^t s\,dW_s}_{=\text{Wiener process Integral}} + \underbrace{\int_0^t W_s\,ds}_{=\text{FV}}$

e) $e^t W_t^2$

$$e^t W_t^2 = \int_0^t e^s \, dW_s^2 + \int_0^t W_s^2 \, de^s + [e^t, W]_t$$

$$= \int_0^t e^s \, d\left(2\int_0^s W_s \, dW_s + s\right) + \int_0^t W_s^2 \, de^s$$

$$= \int_0^t 2 W_s e^s \, dW_s + \int_0^t e^s \, ds + \int_0^t W_s^2 \, de^s$$

$$= (e^t - 1) + \int_0^t 2 W_s e^s \, dW_s + \int_0^t W_s^2 \, de^s$$

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Exam problems:

1) let $X_t = \int_0^t h(s)\,dW_s$ a Wiener integral

$$[X]_t = [X,X]_t = [\int_0^t h(s)\,dW_s, \int_0^t h(s)\,dW_s]_t = \int_0^t h(s)^2\, d\underbrace{[W]_s}_{s}$$

$$= \int_0^t h(s)^2\,ds$$ Theorem 4.4.

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2) a) let $X_t = \int_0^t h_1(s)\,dW_s$, $Y_t = \int_0^t h_2(s)\,dW_s$

$$[X,Y]_t = [\int_0^t h_1(s)\,dW_s, \int_0^t h_2(s)\,dW_s] = \int_0^t h_1(s)h_2(s)\,d\underset{\overset{\shortparallel}{s}}{[W]}$$ (know by heart)

$$= \int_0^t h_1(s)h_2(s)\,ds$$

b) let $X_t = \int_0^t h_1(s)\,dW_s^1$, $Y_t = \int_0^t h_2(s)\,dW_s^2$

$$[\int_0^t h_1(s)\,dW_s^1, \int_0^t h_2(s)\,dW_s^2] = 0$$

3) c) dN_t^2 $\quad N_t^2 = 2\int_0^t N_s\,dN_s + N_t$

$$N_t^2 = \underset{X_t}{\overset{\shortparallel}{N_t}} \cdot \underset{Y_t}{\overset{\shortparallel}{N_t}}$$

$$[X,Y] = \sum \Delta X \Delta Y$$

$$[N]_t$$
$$\shortparallel$$

$$N_t^2 = N_0 N_0 + \int_0^t N_{s-}\,dN_s + \int_0^t N_{s-}\,dN_s + [N,N]_t$$

$$\sum(\Delta N_t) = \sum_{s\le t} \Delta N_s = N_t$$

$$[N,N] = \sum_{s\le t} (\Delta N_s)^2 = \sum \Delta N_s = N_t$$

because it jumps by 1.

5, 6) Integration by parts

$$W_t^2 = \underset{=0}{W_0^2} + 2\int_0^t W_s\,dW_s + \int_0^t W_s\,dW_s + \underset{\underset{t}{[W]_t}}{[W,W]_t}$$

$$W_t^2 = \int_0^t \underset{a_s = 1}{1}\,ds + \int_0^t \underbrace{2W_s}_{b_s}\,dW_s \qquad a_s = 1,\ b_s = 2W_s$$

$$dW_t^2 = 1\,dt + 2W_t\,dW_t$$

↑ drift term ↑ diffusion term

15, ~~B~~,

a) $X_t = W_t \quad f(x) = x^2$

$d(W_t^2) = ?$ Apply Itô's formula $\quad f'(x) = 2x \quad f'' = 2$

$d(f(x)) = f'(x)\,dx + \frac{1}{2} f''(x)\, d[X]_t$

$d(W_t^2) = 2 W_t\, dW_t + \cancel{\tfrac{1}{2} 2 dW_t} + \frac{1}{2} \cdot 2\, d\underbrace{[W]_t}_{=t}$

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$d(W_t^2) = 1 dt + 2 W_t dW_t$

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5, e) $\quad X_t^2 \Rightarrow f(X_t)$ function of X_t

$e^t X_t = f(t, X_t)$ function of t and X_t

$$\underline{d(f(t,X_t))} = \underbrace{\frac{\partial f(t,X_t)}{\partial X}}_{= f'(x)} dx + \frac{\partial f(t,X_t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f(t,X_t)}{\partial X^2} d[X]_t$$

5, e)

$Y_t = e^t W_t^2$

$f(t, X_t) = e^t X_t^2$

$$d(f(t,X_t)) = \underbrace{(2e^t X_t)}_{\frac{\partial f}{\partial x}} dX_t + \underbrace{e^t X_t^2}_{\frac{\partial f}{\partial t}} dt + \frac{1}{2} \cdot \cancel{2} \cdot e^t d[X]_t$$

$= \cancel{(2e^t dW_t}$

$= (2 e^t W_t) dW_t + e^t W_t^2 dt + e^t d[W]_t$

$= 2 e^t W_t dW_t + e^t W_t^2 dt + e^t dt$

$= \underbrace{e^t \cdot (1 + W_t^2)}_{a_t} dt + \underbrace{2 e^t W_t}_{b_t} dW_t$

or Integration by parts

$$e^t W_t^2 = \underbrace{e^0 W_0^2}_{=0} + \int_0^t e^s d(W_s^2) + \int_0^t W_s^2 d(e^s) + \underbrace{[e^t, W^2]_t}_{=0}$$

continuous

[since e^t continuous FV)

~~corrolary~~

corollary 4.8

$= \int_0^t e^s (2W_s\, dW_s + ds) + \int_0^t W_s^2 e^s ds$

$= 2 \int_0^t e^s W_s dW_s + \int_0^t e^s ds + \int_0^t W_s^2 e^s ds$

$d(e^t W_t^2) = e^t dt + e^t W_t^2 dt + 2 e^t W_t dW_t$

$= \underbrace{e^t (1 + W_t^2)}_{a_t} dt + \underbrace{2 e^t W_t}_{b_t} dW_t$

8) let $X_t = \int_0^t W_s\, ds$

a) Show that $X_t = \int_0^t (t-s)\, dW_s$

$$W_t t = \underbrace{\int_0^t W_s\, ds}_{X_t} + \int_0^t s\, dW_s + \overbrace{[t, W]_t}^{=0}$$

Klient $X_t = tW_t - \int_0^t s\, dW_s$ — Mitarbeiter — Seite

Auftrag $X_t = t\int_0^t 1\, dW_s - \int_0^t s\, dW_s$ — Datum — Ablage

$$X_t = \int_0^t t\, dW_s - \int_0^t s\, dW_s = \underline{\underline{\int_0^t t-s\, dW_s}}$$

b) $X_t = \underbrace{\int_0^t (t-s)\, dW_s}$ → normally distributed

or $X_t = tW_t - \int_0^t s\, dW_s$

normal — normal

and show that the covariance between is zero

c) $Cov\left(\int_0^t (s-u)\, dW_u, \int_0^t (t-u)\, dW_u\right)$ $\quad s \leq t$

$$Cov(\quad , \quad) = E\left[\int_0^t (s-u)\, dW_u \cdot \int_0^t (t-u)\, dW_u\right]$$

$$= E\left[\int_0^s (s-u)\, dW_u \left(\int_0^s (t-u)\, dW_u + \int_s^t (t-u)\, dW_u\right)\right] =$$

$$= E\left[\int_0^s (s-u)\, dW_u \int_0^s (t-u)\, dW_u\right] + \underbrace{E\left[\int_0^s (s-u)\, dW_u \cdot \int_s^t (t-u)\, dW_u\right]}$$

$$E\left[E\left[\int_0^s (s-u)\, dW_u \int_0^t (t-u)\, dW_u \mid F_s\right]\right]$$ ~~call $m_R = \int_0^R (t-u)\, dW_u$~~

$$= E\left[\int_0^s (s-u)\, dW_u \cdot \underbrace{E\left[\int_s^t (t-u)\, dW_u \mid F_s\right]}_{=0 \text{ since martingale}}\right]$$

$$= E\left[\int_0^s (s-u)\, dW_u \int_0^s (t-u)\, dW_u\right] \underset{\text{isometric equality}}{=} \int_0^s E(\overbrace{(s-u)\,(t-u)}^{\text{deterministic}}\, du) = \int_0^s st - us - ut + u^2\, du$$

$$= stu - \frac{u^2 s}{2} - \frac{u^2 t}{2} + \frac{u^3}{3}\Big|_0^s = \cancel{s^2 u - \frac{u^2 s}{2} - \frac{u^2 s}{2} + u}\; s^2 t - \frac{s^3}{2} - \frac{s^2 t}{2} + \frac{s^3}{3} = \underline{\underline{\frac{s^3}{3} + \frac{s^2}{2}(t-s)}}$$

$$= \frac{3s^3}{2} + \frac{2s^3}{6}$$

or

$$E\left[\int_0^s (s-u)\,dW_u \int_0^t (t-u)\,dW_u\right] =$$

isometric equality

$$= E\left[\int_0^t 1_{\{u\le s\}} (s-u)\,dW_u \int_0^t (t-u)\,dW_u\right] = \int_0^t I_{\{u\le s\}} (s-u)(t-u)\,du$$

$$= \int_0^s (s-u)(t-u)\,du$$

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14) $dX_t = (\nu - \mu X_t)dt + \sigma dW_t \Rightarrow X_t = X_0 + \int (\nu - \mu X_s)ds + \int \sigma dW_s$

X_0 is random variable

$\Downarrow$ $Y_0 = e^{\mu \cdot 0} X_0 = X_0$

$\cancel{X_t}$ $dM_t = M_t dt$ (if $M_0 = 1$)

$M_t = e^t$

$e^{-t} M_t = e^t \cdot e^{-t}$

$Y_t = e^{\mu t} X_t$

$\cancel{dY_t = e^{\mu t} dX_t + \mu e^{\mu t} d}$

$dY_t = e^{\mu t} dX_t + X_t \mu e^{\mu t} dt + 0$ quadratic covariation

$dY_t = e^{\mu t}((\nu - \mu X_t)dt + \sigma dW_t) + \mu e^{\mu t} X_t dt$

$= e^{\mu t}\nu dt - \cancel{e^{\mu t}\mu X_t t} + e^{\mu t}\sigma dW_t + \cancel{\mu e^{\mu t} X_t} dt$

$dY_t = e^{\mu t}\nu dt + e^{\mu t}\sigma dW_t$

$Y_t = Y_0 + \nu \int_0^t e^{\mu s} ds + \sigma \int_0^t e^{\mu s} dW_s$

$Y_0 = X_0$

$Y_t = X_0 + \frac{\nu}{\mu}(e^{\mu t} - 1) + \sigma \int_0^t e^{\mu s} dW_s$

We have $X_t = e^{-\mu t} Y_t$

$$\boxed{X_t = e^{-\mu t} X_0 + \frac{\nu}{\mu} - \frac{\nu}{\mu} e^{-\mu t} + \sigma \int_0^t e^{\mu (s-t)} dW_s}$$

$\int E[e^{2\mu(s-t)}] ds < \infty$

$\Rightarrow$ martingale

i) $X_0 = x_0$

$E[X_t] = e^{-\mu t} x_0 + \frac{\nu}{\mu}(1 - e^{-\mu t}) + 0$

$\lim_{t\to\infty} E[X_t] = \frac{\nu}{\mu}$

(8)

$$Var[X_t] = E[(X_t - E[X_t])^2] = E[(\sigma \int_0^t e^{\mu(s-t)} dW_s)^2] \underset{\text{isometry}}{=} \sigma^2 \int_0^t e^{2\mu(s-t)} ds =$$

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$$= \frac{\sigma^2 (1 - e^{-2\mu t})}{2\mu}$$

$$\lim_{t\to\infty} Var(X_t) = \frac{\sigma^2}{2\mu}$$

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b) $Cov(X_s, X_t) = \cancel{E[(X_t - E(X_t)]} \; E[(X_t - E(X_t))(X_s - E(X_s))] =$

$$= E[e^{-\mu s}\sigma \int_0^s e^{\mu u} dW_u \; e^{-\mu t}\sigma \int_0^t e^{\mu u} dW_u] =$$

$$= \sigma^2 \cdot e^{-\mu(s+t)} E[\int_0^s e^{\mu u} dW_u \int_0^t e^{\mu u} dW_u]$$ $\Rightarrow$ use Indicator Function and isometric equality

$$= \cancel{2}\sigma^2 \cdot e^{-\mu(s+t)} \int_0^s e^{2\mu u} du$$

$$= \frac{\sigma^2}{2\mu} \cdot (e^{-\mu(t-s)} - e^{-\mu(s+t)})$$

write $\int_0^s e^{\mu u} dW_u = \int_0^t I_{\{u \le s\}} e^{\mu u} dW_u$

$$\lim_{t\to\infty} = 0$$

$\Rightarrow$ Now suppose X_0 is a random variable, $X_0 \sim N(\frac{\nu}{\mu}, \frac{\sigma^2}{2\mu})$

$$E[X_t] = e^{-\mu t} \underbrace{E[X_0]}_{\frac{\nu}{\mu}} + \frac{\nu}{\mu} - e^{-\mu t}\frac{\nu}{\mu} = e^{-\mu t}\frac{\nu}{\mu} + \frac{\nu}{\mu} - e^{-\mu t}\frac{\nu}{\mu} = \frac{\nu}{\mu}$$

$Var(X_t) = E\cancel{[(X_t - E(X_t))^2] = E[(e^{-\mu t} X_0 - e^{-\mu t}}$

$$X_t = e^{-\mu t} X_0 + \frac{\nu}{\mu}(1 - e^{-\mu t}) + e^{-\mu t}\sigma \int_0^t e^{\mu s} dW_s$$

$Var(X_t) = Var(e^{-\mu t} X_0) + Y +$

$\cancel{Cov(e^{-\mu t} X_0, \frac{\nu}{\mu}(1 - e^{-\mu t})}$

$Cov(\underbrace{e^{-\mu t} X_0}_{I}, \sigma^2 \underbrace{\int_0^t e^{\mu(s-t)} dW_s}_{II}) = E[e^{-\mu t} X_0 \sigma^2 \int \ldots] \ldots$

$E[I \; II] - E[I] E[II]$

$$Var(X_t) = Var(e^{-\mu t} X_0) + Y + 0$$

$$= e^{-\mu t} var(X_0) +$$

$$= \frac{\sigma^2 \cdot (1 - e^{-2\mu t})}{2\mu} + \frac{\sigma^2}{2\mu} = \frac{\sigma^2}{2\mu}$$

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$\int_0^t f_s g_s ds \qquad \int_0^t f_s dg_s = \int_0^t f_s g'_s ds = f(t)g(t) - f(0)g(0) - \int_0^t f'_s g_s ds$

$f \quad g'$
$f' \quad g$

$\int_0^t f'_s g_s ds = \int_0^t g_s df_s$

$\int_0^t f dg = f_t g_t - f_0 g_0 - \int_0^t g_s df_s + \underbrace{[g, f]_t}_{=0}$

$W_t^2 = 2\int_0^t W_s dW_s + t$

$W_t^2 = \underbrace{W_0^2}_{=0} + \int_0^t W_s dW_s + \int_0^t W_s dW_s + \underbrace{[W, W]_t}_{[W] = t}$

$W_t^2 = 2\int_0^t W_s dW_s + t$

$dW_t^2 = W_t^2 = 2\int_0^t W_s dW_s + t$

$dW_t^3 = W_t^2 \cdot W_t = \int_0^t W_s^2 dW_s + \int_0^t W_s \, dW_s^2 + [W^2, W]_t$

ds